Problems in laser physics

Sheet 6

Handed out on 7. 12. 17 for the Tutorial on 11. 1. 18

Problem 16: Hermite-Gaussian beams (2P)

In the case of a confocal resonator with rectangular mirrors, the electric field distribution of the eigenmodes is given by the Hermite-Gaussian modes. Their radially Gaussian shape is modulated by the Hermite polynomials $H_p(x)$, which are given by the relation

$$H_m(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m}(e^{-x^2}) \ . \tag{1}$$

(a) Calculate the first three Hermite polynomials $H_0(x)$, $H_1(x)$ and $H_2(x)$ (1P).
(b) The Hermite polynolmials are orthogonal under the integral weight e^{-x^2}, i.e.

$$\forall m \neq n \Rightarrow \int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = 0 \ . \tag{2}$$

Show this orthogonality explicitly for the combinations $(m, n) = (0, 1)$, $(1, 1)$ and $(1, 2)$, exploiting the symmetry of the integrands (1P).

Problem 17: Etalon (5P)

An etalon is a plane-parallel plate of an optical medium with refractive index n and thickness d, which is used as an intracavity element for mode selection and fine tuning of lasers. The frequency selectivity of an etalon is often enhanced by a dielectric coating on both sides, causing a reflectivity of R for each side. Assuming a perpendicular incidence of a laser beam, or a laser beam which is much larger than the thickness of the etalon, with an electric field amplitude E_0 onto the plate, we obtain multiple reflections from the front and the back side of the etalon. The first reflection is created by the entrance surface, and as we look at the electric field amplitude, this reflected field is

$$|E_1| = \sqrt{R}|E_0| \; . \tag{3}$$

The transmitted light $\sqrt{1-R}|E_0|$ is partially reflected back by the exit surface, resulting in an incident electric field onto the first surface of

$$|E_2'| = \sqrt{R(1-R)}|E_0| \; . \tag{4}$$

From this field the fraction

$$|E_2| = \sqrt{1-R}|E_2'| = \sqrt{R}(1-R)|E_0| \tag{5}$$

is transmitted at the front surface and adds to the reflected field of the first reflection E_1. However, in this addition we have to take into account the phase difference between the first reflection E_1 and the field E_2

$$\Delta\varphi = 2\pi\frac{\Delta s}{\lambda} \; , \tag{6}$$

which is given by the additional optical path length of the etalon

$$\Delta s = 2d\sqrt{n^2 - \sin^2\alpha} \; . \tag{7}$$

Therein, α is the angle of incidence of the beam onto the etalon. Thus, assuming an infinite number of back and forth reflections, the total reflected field amplitude from the etalon is given by

$$E_R = E_0\sqrt{R}\left[1 - (1-R)e^{i\Delta\varphi}\sum_{m=0}^{\infty} R^m e^{im\Delta\varphi}\right] \; . \tag{8}$$

(a) Calculate the total reflected field amplitude E_R and show that the reflected and transmitted intensities I_R and I_T of a lossless etalon are given by the **Airy formulas**

$$I_R = I_0\frac{A\sin^2\frac{\Delta\varphi}{2}}{1 + A\sin^2\frac{\Delta\varphi}{2}} \quad \text{and} \quad I_T = I_0\frac{1}{1 + A\sin^2\frac{\Delta\varphi}{2}} \; , \tag{9}$$

with $A = \frac{4R}{(1-R)^2}$. (3P)
(b) Show that the maxima of transmission occur for the wavelengths $\lambda_{max,j} = \frac{2d}{j}\sqrt{n^2 - \sin^2\alpha}$. (1P)
(c) Show that the minima of transmission occur for the wavelengths $\lambda_{min,j} = \frac{4d}{2j+1}\sqrt{n^2 - \sin^2\alpha}$. (1P)

Problem 18: Longitudinal modes and laser line width (3P)

(a) Calculate the longitudinal mode index for a $632.8\ \mathrm{nm}$ HeNe laser with a $300\ \mathrm{mm}$ long confocal cavity operating on its fundamental mode (1P).
(b) Calculate the spontaneously emitted power into this mode for a total laser output power of $10\ \mathrm{mW}$ at an OC reflectivity of 98%, assuming a four-level laser scheme. (1P).
(c) Calculate the theoretical linewidth of the laser in (b), assuming 0.4% internal losses (1P).